Resit Exam

10/04/2024, 8:30 am - 10:30 am

Instructions:

- Prepare your solutions in an **ordered, clear and clean way**. Avoid delivering solutions with scratches.
- Write your name and student number in **all** pages of your solutions.
- Clearly indicate each exercise and the corresponding answer. Provide your solutions with as much detail as possible.
- Use different pieces of paper for solutions of different exercises.
- Read first the whole exam, and make a strategy for which exercises you attempt first. Start with those you feel comfortable with! The exam has a total value of 13 points.

Exercise 1: Let $f:\mathbb{R}^2\to\mathbb{R}$ be defined by

$$f(x,y)=\begin{cases}0, & (x,y)=(0,0)\\ \dfrac{xy(x^2-y^2)}{x^2+y^2}, & (x,y)\neq(0,0).\end{cases}$$

(a) (0.5 points) Do all directional derivatives of f at $(x,y)=(0,0)$ exist? Evaluate the directional derivatives whenever they exist.

(b) (0.5 points) Calculate $\mathrm{D}_1 f$ and $\mathrm{D}_2 f$ at $(x,y)\neq(0,0)$.

(c) (0.5 points) Show that $\mathrm{D}_1 f$ and $\mathrm{D}_2 f$ exist at $(x,y)=(0,0)$.

(d) (1 point) Is f differentiable at $(x,y)=(0,0)$? Justify your answer.

(e) (0.5 points) Is f continuous at $(x,y)=(0,0)$? Justify your answer.

(f) (0.5 points) Show that $\mathrm{D}_2\mathrm{D}_1 f$ and $\mathrm{D}_1\mathrm{D}_2 f$ exist at $(x,y)=(0,0)$, but that they are not equal there. What does this tell us about the differentiability class of f?

Exercise 2: (1.5 points) Let $g:\mathbb{R}^2\to\mathbb{R}^2$ be defined by the equation $g(x,y)=(x,y+x^2)$. Let $f:\mathbb{R}^2\to\mathbb{R}$ be defined as

$$f(x,y)=\begin{cases}0, & (x,y)=(0,0),\\ \dfrac{x^2y}{x^4+y^2}, & (x,y)\neq(0,0).\end{cases}$$

Let $h=f\circ g$. Show that the directional derivatives of f and g exist everywhere, but that there is a vector $\vec{\mathbf{u}}\neq\mathbf{0}$ for which the directional derivative of h in the direction of $\vec{\mathbf{u}}$, that is $\mathrm{D}h(\mathbf{0})\vec{\mathbf{u}}$, does not exist.

Exercise 3: Let $f:\mathbb{R}^2\to\mathbb{R}^2$ be defined by the equation

$$f(x,y)=\left(x^2-y^2,2xy\right).$$

Let $A=\left\{(x,y)\in\mathbb{R}^2\,|\,x>0\right\}$.

(a) (0.5 points) Show that f is one-to-one on the set A. [Hint: you may use the fact that if $f(x,y)=f(a,b)$, then $\|f(x,y)\|=\|f(a,b)\|$ (where $\|\cdot\|$ denotes the Euclidean norm) to find a contradiction.]

(b) (1 point) If g is the inverse function, find $\mathrm{D}g(0,1)$.

Exercise 4: (1 point) Let $f:\mathbb{R}^{k+n}\to\mathbb{R}^n$ be of class C^1; suppose that $f(\mathbf{a})=\mathbf{0}$ and that $Df(\mathbf{a})$ has rank n. Show that if $\mathbf{c}$ is a point of $\mathbf{R}^n$ sufficiently close to 0 , then the equation $f(\mathbf{x})=\mathbf{c}$ has a solution.

Exercise 5: (0.5 points) Let S be a closed curve in $\mathbb{R}^2$ and C the unit circle in $\mathbb{R}^2$. Suppose that S and C are diffeomorphic. What is the 2-dimensional volume of the curve S ($\mathrm{vol}_2\, S$)? Justify your answer in full detail.

Hints and remarks: for this exercise you may assume that "S and C are diffeomorphic" means that there is a C^r-function f, $r\geq 1$, with C^r inverse, such that $f:S\to C$ and $f^{-1}:C\to S$.

Exercise 6: (1 point) Let B be the portion of the first quadrant in $\mathbb{R}^2$ lying between the hyperbolas $xy=1$ and $xy=2$ and the two straight lines $y=x$ and $y=4x$. Evaluate $\int_B x^2y^3\,dx dy$. [Hint: use the change of variables $x=u/v$ and $y=uv$.]

Exercise 7: Let $U = \mathbb{R}^2 \backslash \mathbf{0}$; consider the 1-form in A defined by the equation

$$\omega = \frac{x\,\mathrm{d}x + y\,\mathrm{d}y}{x^2 + y^2}.$$

(a) (1 point) Show that ω is closed.

(b) (1 point) Show that ω is exact on U.

Exercise 8: (1 point) Let C be a closed curve in the plane. Show that $\begin{bmatrix} xy^2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -x^2 y \end{bmatrix}$ do equal work around C.

Exercise 9: (1 point) What is the integral $\int_S x_3 \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_4$, where S is the part of the three-dimensional manifold of equation

$$x_4 = x_1 x_2 x_3 \qquad \text{where } 0 \le x_1, x_2, x_3 \le 1,$$

oriented by $\Omega = \operatorname{sgn} \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3$? [Hint: This surface is a graph, so it is easy to parametrize.]